REFLECTION OF MAGNETOACOUSTIC WAVES

L. Ya. Kosachevskii

The problem of the reflection of magnetoacoustic waves at the boundary dividing an elastic medium from a fluid medium with infinite conductivity in the presence of an arbitrary constant magnetic field was treated in [1]. In writing down the boundary conditions the continuity of the tangential component of the magnetic field was used. This condition is valid when the conductivity of the medium is finite but not when the conductivity is infinite. In this connection a problem similar to that in [1] is solved, without employing this particular boundary condition. The amplitude conversion coefficients found for the limiting cases of weak and strong magnetic fields coincide with the respective coefficients given in [2,3] for media with a finite conductivity, if we allow the conductivity in the latter expressions to become infinite.

1. The linearized equations describing the propagation of perturbations in a continuous medium with finite conductivity have the form [4]

$$\rho\frac{\partial v_i}{\partial t}=\frac{\partial}{\partial x_k}(P_{ik}+T_{ik}),\qquad \frac{\partial\rho}{\partial t}+\rho\,\mathrm{div}\,\mathbf{v}=0$$
$$\mathrm{rot}\,\mathbf{E}=-c^{-1}\partial\mathbf{h}/\partial t,\qquad \mathrm{div}\,\mathbf{h}=0,\qquad \mathbf{E}=-c^{-1}\mathbf{v}\times\mathbf{H}. \tag{1.1}$$

Here **H** is the external magnetic field, assumed to be constant; h is a small change in the magnetic field of the wave; **E** is the induced electric field; ρ and v are the density and velocity of the medium; c is the velocity of light; P_{ik} is the stress tensor; T_{ik} is the Maxwell stress tensor which assumes the following form after linearization:

$$T_{ik}=\tfrac{1}{4}\pi^{-1}[H_iH_k+H_ih_k+H_kh_i-\tfrac{1}{2}(H^2+2H_lh_l)\delta_{ik}].$$

At the boundary dividing the two media the first four equations of (1.1) correspond to the boundary conditions [5]

$$[P_{ik}n_k+T_{ik}n_k]=0,\quad [v_in_i]=0,\quad [E_i\tau_i]=0,\quad [h_in_i]=0. \tag{1.2}$$

Here n and τ are unit vectors normal and tangential to the boundary, while the brackets denote the discontinuity of the value at the boundary. It follows from the fifth equation of (1.1) together with the second and third conditions of (1.2) that the tangential component of the velocity vector is continuous at the boundary

$$[v_i\tau_i]=0. \tag{1.3}$$

2. In the case of plane waves in a fluid, Eq. (1.1) reduce to the system of algebraic equations

$$\omega\mathbf{v}=\frac{p}{\rho_0}\mathbf{k}+\frac{1}{4\pi\rho_0}\mathbf{H}\times(\mathbf{k}\times\mathbf{h})$$
$$\omega p=\rho_0a_0^2\mathbf{k}\cdot\mathbf{v},\quad \omega\mathbf{h}=-\mathbf{k}\times(\mathbf{v}\times\mathbf{H}),\quad P_{ik}=-p\delta_{ik} \tag{2.1}$$

where ρ_0 and a_0 are the density and velocity of sound in the fluid, and p is the hydrodynamic pressure. We neglect the viscosity of the medium.

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We assume that the wave vector **k** and the vector H lie in the xz plane. The dispersion equation for waves polarized in this plane follows from (2.1):

$$u^2-(1+\psi_0)u+\psi_0\cos^2\alpha=0, \quad u=\left(\frac{\omega}{ka_0}\right)^2, \quad \psi_0=\frac{H^2}{4\pi\rho_0 a_0^2}. \tag{2.2}$$

Here u and ψ_0 are the squares of the phase velocity and magnetic field strength in dimensionless form, α is the angle between the vectors k and H. The two roots u_1 and u_2 of Eq. (2.2) correspond to the fast and slow magnetoacoustic waves.

The following relations are also obtained from (2.1):

$$\begin{gathered} V_x=Mv_z,\ h_x=Av_z,\ E_y=Bv_z,\ -p=Zv_z \\ M=\beta^{-1}(k_z\cos\alpha-ku\sin\varphi),\ \beta=ku\cos\varphi-k_x\cos\alpha \\ A=\frac{k_z}{ka_0}u^{-1/2}(H_x-MH_z), \quad B=-\frac{\omega}{ck_z}A, \quad Z=-\frac{\rho_0 a_0}{k}u^{-1/2}(k_xM+k_z) \end{gathered} \tag{2.3}$$

where φ is the angle of inclination of the magnetic field to the x axis.

3. The propagation of waves in an unbounded elastic conducting medium was treated in papers [6,7].

In an elastic medium

$$P_{ik}=\lambda u_{ll}\delta_{ik}+2\mu u_{ik}, \qquad u_{ik}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)$$

where λ and μ are the Lamé coefficients.

Remembering that in the case of plane monochromatic waves the displacement vector **u** is related to the velocity vector by the relation

$$\mathbf{u}=-\frac{1}{i\omega}\mathbf{v}$$

and taking (1.1) into account, we have

$$\begin{gathered} \omega^2\mathbf{v}=a^2\mathbf{k}(\mathbf{kv})-b^2\mathbf{k}\times(\mathbf{k}\times\mathbf{v})+\frac{\omega}{4\pi\rho}\mathbf{H}\times(\mathbf{k}\times\mathbf{h}) \\ \omega\mathbf{h}=-\mathbf{k}\times(\mathbf{v}\times\mathbf{H}), \quad a^2=\frac{\lambda+2\mu}{\rho}, \quad b^2=\frac{\mu}{\rho}. \end{gathered} \tag{3.1}$$

Here a and b are the velocities of purely elastic longitudinal and transverse waves, and ρ is the density of the elastic medium.

The following dispersion equation can be obtained from (3.1) for waves polarized in the xz plane:

$$\begin{gathered} u^2-(1+\xi+\psi)u+\xi+\psi(\cos^2\alpha+\xi\sin^2\alpha)=0 \\ u=\left(\frac{\omega}{ka}\right)^2, \quad \psi=\frac{H^2}{4\pi\rho a^2}, \quad \xi=\left(\frac{b}{a}\right)^2. \end{gathered} \tag{3.2}$$

The roots u_3 and u_4 of this equation correspond to the fast and slow magnetoacoustic waves.

In accordance with (3.1) we have the following relations for waves in the elastic medium:

$$\begin{gathered} v_x=Mv_z,\ h_x=Av_z,\ E_y=Bv_z,\ P_{zz}=Zv_z,\ P_{xz}=Xv_z \\ M=\beta^{-1}[(1-\xi)k_z\cos\alpha-k(u-\xi)\sin\varphi] \\ \beta=k(u-\xi)\cos\varphi-(1-\xi)k_x\cos\alpha \\ A=\frac{k_z}{ka}u^{-1/2}(H_x-MH_z), \qquad B=-\frac{\omega}{ck_z}A \end{gathered}$$

$$Z = -\frac{\rho a}{k} u^{-1/2} [k_z + (1 - 2\xi) k_x M], \qquad X = -\frac{\xi \rho a}{k} u^{-1/2} (k_x + k_z M) \; . \tag{3.3}$$

4. Let us assume that the boundaries of the fluid and elastic medium coincide with the xy plane.

From (1.2) and (1.3), the boundary conditions for waves polarized in the xz plane assume the form

$$\left[P_{zz} - \frac{1}{4\pi} H_x h_x\right] = 0, \quad \left[P_{xz} + \frac{1}{4\pi} H_z h_x\right] = 0, \quad [v_z] = 0, \quad [v_x] = 0 \, . \tag{4.1}$$

The waves which are polarized perpendicular to the xz plane (Alfven waves), propagate independently and will not be treated here.

Let a fast magnetoacoustic wave be incident on the boundary from the side of the fluid (Fig. 1). The incident wave excites a system of four waves at the boundary; two magnetoacoustic waves in the fluid and two magnetoelastic waves in the elastic medium. Quantities referring to the incident perturbations will be denoted by primes. Taking the amplitude v_{1z}' to be unity,we write down the velocity field in the fluid

$$v_z = -\exp\{-i[\omega t - (k_1' r)]\} + \sum_{\nu=1}^{2} W_\nu \exp\{-i[\omega t - (k_\nu r)]\}$$

and in the elastic medium

$$v_z = -\sum_{\nu=3}^{4} W_\nu \exp\{-i[\omega t - (k_\nu r)]\} \, ,$$

Here W_ν are the amplitude conversion coefficients.

We can obtain a system of equations for determining these coefficients from conditions (4.1) taking (2.3) and (3.3) into account.

$$\begin{gathered}\sum_{\nu=1}^{4} W_\nu = 1, \qquad \sum_{\nu=1}^{4}\left(Z_\nu - \frac{H_x}{4\pi} A_\nu\right) W_\nu = Z_1' - \frac{H_x}{4\pi} A_1' \\ \sum_{\nu=1}^{4} M_\nu W_\nu = M_1', \qquad \sum_{\nu=1}^{4}\left(X_\nu + \frac{H_z}{4\pi} A_\nu\right) W_\nu^* = \frac{H_z}{4\pi} A_1', \qquad X_1 = X_2 = 0 \, .\end{gathered} \tag{4.2}$$

The solution of system (4.2) has the form

$$\begin{aligned} W_1 &= \Delta^{-1}\{\xi[(M_2 - M_1') N + mQ_1'] + \psi \sin^2\varphi \, (P_1' + mR_1')\} \\ W_2 &= \Delta^{-1}\{\xi[(M_1' - M_1) N + mQ_2'] + \psi \sin^2\varphi \, (P_2' + mR_2')\} \\ W_3 &= (M_4 - M_3)^{-1}[M_4 - M_1' - (M_4 - M_1) W_1 - (M_4 - M_2) W_2] \\ W_4 &= (M_3 - M_4)^{-1}[M_3 - M_1' - (M_3 - M_1) W_1 - (M_3 - M_2) W_2] \, . \end{aligned} \tag{4.3}$$

Here

$$\Delta = \xi[(M_2 - M_1) N + mQ] + \psi \sin^2\varphi \, (P + mR)$$

$$\begin{aligned} N = (1 - M_4 \operatorname{ctg}\theta_4)&[(1 - 2\xi) M_3 - \operatorname{ctg}\theta_3] \\ &- (1 - M_3 \operatorname{ctg}\theta_3)[(1 - 2\xi) M_4 - \operatorname{ctg}\theta_4] \end{aligned}$$

$$\begin{aligned} Q = [M_4 - M_3 + M_3 M_4 (\operatorname{ctg}\theta_4 - \operatorname{ctg}\theta_3)]&(M_2 - M_1 + \operatorname{ctg}\theta_2 - \operatorname{ctg}\theta_1) \\ &+ (M_4 \operatorname{ctg}\theta_4 - M_3 \operatorname{ctg}\theta_3)(M_2 \operatorname{ctg}\theta_1 - M_1 \operatorname{ctg}\theta_2) \end{aligned}$$

$$R = (M_2 + \operatorname{ctg}\theta_2)\gamma_1 - (M_1 + \operatorname{ctg}\theta_1)\gamma_2, \quad P = \Gamma_3\gamma_4 - \Gamma_4\gamma_3$$

$$\begin{aligned} \gamma_\nu = (M_4 - M_3)(\operatorname{ctg}\varphi - M_\nu) \operatorname{ctg}\theta_\nu &+ (M_4 - M_\nu)(\operatorname{ctg}\varphi - M_3) \cdot \operatorname{ctg}\theta_3 \\ &+ (M_\nu - M_3)(\operatorname{ctg}\varphi - M_4) \operatorname{ctg}\theta_4 \qquad (\nu = 1{,}2) \end{aligned}$$

$$\gamma_\nu = (M_2 - M_1)(\mathrm{ctg}\,\varphi - M_\nu)\,\mathrm{ctg}\,\theta_\nu + (M_2 - M_\nu)(\mathrm{ctg}\,\varphi - M_1)\,\mathrm{ctg}\,\theta_1 + (M_\nu - M_1)(\mathrm{ctg}\,\varphi - M_2)\,\mathrm{ctg}\,\theta_2 \quad (\nu = 3,4)$$

$$\Gamma_\nu = \xi(1 - M_\nu\,\mathrm{ctg}\,\theta_\nu)\,\mathrm{ctg}\,\varphi + (1-2\xi) M_\nu - \mathrm{ctg}\,\theta_\nu, \quad m = \rho_0 a_0^2/\rho a^2 .$$

The quantities P_V, $Q_{V'}$, $R_{V'}$ are obtained from P, Q, and R, respectively if M_ν is changed to M_1' in the latter and ctg θ_ν replaced by ctg θ_1'.

For a weak magnetic field ($\psi_0 \ll 1$, $\psi \ll 1$) we have, with an accuracy to the dominant terms,

$$\theta_1' = \theta_1, \quad \sin\theta_1 = \frac{\sin\theta_2}{\sqrt{u_2}} = \frac{a_0}{a}\sin\theta_3 = \frac{a_0}{b}\sin\theta_4 \tag{4.4}$$

$$u_1 = 1 + \psi_0\sin^2\alpha_1, \quad u_2 = \psi_2\cos^2\alpha_2 = \psi_0\sin^2\varphi\,(1 + 2\sqrt{\psi_0}\cos\varphi\sin\theta_1)$$
$$u_3 = 1 + \psi\sin^2\alpha_3, \quad u_4 = \xi + \psi\cos^2\alpha_4, \quad M_1 = \mathrm{tg}\,\theta_1$$
$$M_2 = -\,\mathrm{ctg}\,\theta_2 - \mathrm{ctg}\,\varphi/\sin^2\theta_1, \quad M_3 = -\,\mathrm{tg}\,\theta_3, \quad M_4 = \mathrm{ctg}\,\theta_4,$$
$$M_1' = -M_1 .$$

The conversion coefficients assume the form

$$W_1 = W_1^\circ - \frac{\sqrt{\psi_0 Y^2}}{2\sin\varphi}\cos\theta_1, \quad W_3 = (1 - W_1^\circ)\cos 2\theta_4 + \frac{\sqrt{\psi}YF}{2\sin\varphi}\left(\frac{\rho_0}{\rho}\right)^{1/2}\cos\theta_3$$

$$W_2 = \sqrt{\psi_0}Y\sin\theta_1, \quad W_4 = 2(1 - W_1^\circ)\sin^2\theta_4 - \frac{\sqrt{\psi}Y\Phi}{2\sin\varphi}\left(\frac{\rho_0}{\rho}\right)^{1/2}\sin\theta_3$$

$$Y = \frac{2\sin\varphi}{Z_n + Z_1^\circ}\left[Z_n\,\mathrm{tg}\,\theta_1 + Z_1^\circ\frac{\sin(2\theta_4 - \theta_3)}{\cos\theta_3}\right], \quad Z_1^\circ = \frac{\rho_0 a_0}{\cos\theta_1}$$

$$F = \frac{2\sin\varphi}{Z_n + Z_1^\circ}\left[Z_1^\circ\,\mathrm{tg}\,\theta_3 + Z_3^\circ\frac{\sin(2\theta_4 - \theta_1)}{\cos\theta_1}\right], \quad Z_3^\circ = \frac{\rho a}{\cos\theta_3} \tag{4.5}$$

$$\Phi = \frac{2\sin\varphi}{Z_n + Z_1^\circ}(Z_1^\circ + Z_3^\circ\cos 2\theta_4 + Z_4^\circ\sin 2\theta_4\,\mathrm{tg}\,\theta_1), \quad Z_4^\circ = \frac{\rho b}{\cos\theta_4}$$

$$W_1^\circ = \frac{Z_n - Z_1^\circ}{Z_n + Z_1^\circ}, \quad Z_n = Z_3^\circ\cos^2 2\theta_4 + Z_4^\circ\sin^2 2\theta_4 .$$

The quantities W_1^0 and Z_n are the reflection coefficient and the total acoustic impedance of the boundary in the absence of a magnetic field respectively [8].

Expressions for the coefficients W_ν were obtained in paper [2] for media with a finite conductivity in the presence of a weak magnetic field. If we allow the conductivity to become infinite in these expressions we obtain Eqs. (4.5).

In the opposite limiting case of a strong magnetic field ($\psi_0 \gg 1$, $\psi \gg 1$)

$$\theta_1' = \theta_1, \quad \sin\theta_3 = \left(\frac{\rho_0}{\rho}\right)^{1/2}\sin\theta_1, \quad \mathrm{ctg}\,\theta_2 = \frac{\sqrt{\psi_0}}{\sin\theta_1\sin\varphi} - \mathrm{ctg}\,\varphi$$
$$\mathrm{ctg}\,\theta_4 = \frac{1}{\sin\theta_3\sin\varphi}\left(\frac{\psi}{1 + \xi\,\mathrm{ctg}^2\varphi}\right)^{1/2} + \frac{(1-\xi)\,\mathrm{ctg}\,\varphi}{1 + \xi\,\mathrm{ctg}^2\varphi} \tag{4.6}$$

$$u_1 = \psi_0 + \sin^2\alpha_1, \quad u_2 = \cos^2\alpha_2 - {}^1/_4\,\psi_0^{-1}\sin^2 2\alpha_2$$
$$u_3 = \psi + \sin^2\alpha_3 + \xi\cos^2\alpha_3, \quad M_1' = M_1 = M_3 = -\,\mathrm{tg}\,\varphi$$
$$u_4 = \cos^2\alpha_4 + \xi\sin^2\alpha_4 - {}^1/_4\,\psi^{-1}(1-\xi)^2\sin^2 2\alpha_4, \quad M_2 = M_4 = \mathrm{ctg}\,\varphi .$$

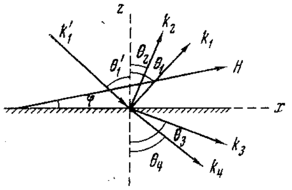

Fig. 1

In this case Eqs. (4.3) give the following expressions, accurate to terms of $1/\psi$:

$$W_1 = \frac{n_3 - n_1}{n_3 + n_1}, \quad W_3 = \frac{2n_1}{n_3 + n_1} \tag{4.7}$$

$$W_2 = \frac{2n_1\rho a\,[(1 - \xi - m)\cos(\theta_3 - \varphi)\sin\varphi - \xi\sin\theta_3]}{\sqrt{\psi}(n_3 + n_1)(\rho a\sqrt{1 + \xi\,\mathrm{ctg}^2\varphi} + \rho_0 a_0)\cos\varphi}$$

$$W_4 = -W_2, \quad n_1 = \rho_0^{1/2}\cos\theta_1, \quad n_3 = \rho^{1/2}\cos\theta_3 .$$

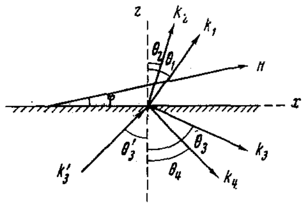

Fig. 2

If we set ξ equal to zero here, we obtain expressions for the reflection coefficients at the boundary of two fluid media, which coincide with the corresponding equations of paper [3].

For the case in which the slow magnetoacoustic wave is incident on the boundary, the coefficients W_ν are found from expressions (4.3) by reversing the indices 1 and 2.

5. Let the fast magnetoelastic wave be incident on the boundary of the fluid from the elastic medium (Fig. 2).

Proceeding as before we find the following expressions for the amplitude coefficients from the boundary conditions:

$$\begin{aligned} W_1 &= (M_2 - M_1)^{-1}[M_2 - M_3' - (M_2 - M_3)W_3 - (M_2 - M_4)W_4] \\ W_2 &= (M_1 - M_2)^{-1}[M_1 - M_3' - (M_1 - M_3)W_3 - (M_1 - M_4)W_4] \\ W_3 &= \Delta^{-1}\{\xi[(M_2 - M_1)N_3' + mQ_3'] + \psi\sin^2\varphi\,(P_3' + mR_3')\} \\ W_4 &= \Delta^{-1}\{\xi[(M_2 - M_1)N_4' + mQ_4'] + \psi\sin^2\varphi\,(P_4' + mR_4')\}. \end{aligned} \tag{5.1}$$

Here N_ν', P_ν', Q_ν', R_ν' result from N, P, Q, R, respectively when M_ν is replaced by M_3' and $\mathrm{ctg}\,\theta_\nu$ by $\mathrm{ctg}\,\theta_3'$.

For a weak magnetic field we have from (5.1)

$$\begin{aligned} W_1 &= (1 - W_3^\circ)\sec 2\theta_4 + \frac{\sqrt{\psi_0}FY}{2\sin\varphi}\cos\theta_1, \qquad W_2 = \sqrt{\psi_0}F\sin\theta_1 \\ W_3 &= W_3^\circ - \frac{\sqrt{\psi}F^2}{2\sin\varphi}\left(\frac{\rho_0}{\rho}\right)^{1/2}\cos\theta_3 \\ W_4 &= -2(1 - W_3^\circ)\frac{\sin^2\theta_4}{\cos 2\theta_4} + \frac{\sqrt{\psi}F\Phi}{2\sin\varphi}\left(\frac{\rho_0}{\rho}\right)^{1/2}\sin\theta_3 \\ W_3^\circ &= \frac{1}{Z_n + Z_1^\circ}(Z_1^\circ - Z_3^\circ\cos^2 2\theta_4 + Z_4^\circ\sin^2 2\theta_4) \end{aligned} \tag{5.2}$$

where W_3^0 is the reflection coefficient for a purely elastic longitudinal wave [8].

For a strong magnetic field Eqs. (5.1) give

$$\begin{aligned} W_1 &= \frac{2n_3}{n_1 + n_3}, \qquad W_3 = \frac{n_1 - n_3}{n_1 + n_3}, \qquad W_2 = -W_4 \\ W_4 &= \frac{2n_3\rho a^2[1 - \xi - m)\cos(\theta_1 + \varphi)\sin\varphi - \xi\sin\theta_1]}{a_0\sqrt{\psi_0}(n_1 + n_3)(\rho a\sqrt{1 + \xi\,\mathrm{ctg}^2\varphi} + \rho_0 a_0)\cos\varphi}. \end{aligned} \tag{5.3}$$

If a slow magnetoacoustic wave is incident on the boundary, expressions for the coefficients W_ν can be obtained from (5.1) by reversing the indices 3 and 4.

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